SOME QUESTIONS CONCERNING THE THEORY OF CONICAL FLOW

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Under consideration are the boundary-value problems of the theory of rotational conical flows and a precise scheme for the flow about a triangular wing. Several types of singular points are discussed. These arise with the joining of irrotational and rotational conical flows behind decaying shock waves.

1. For definiteness we will consider a plane triangular wing at an angle of attack δ without side slip in the flow of nonviscous gas which has velocity W_1, Mach number $M_1 > 1$ and sound velocity a_1 (Fig. 1).

We will assume the edge of the wing to be in supersonic; since the conical flows which arise both above and below the wing do not interact they can be considered separately. In conical flows the velocity components, u, v, w, the entropy S, and pressure p depend on the angular variables which we take to be $\xi = x/z$, $\eta = y/z$.

The plane ξ, η has the simple physical meaning that in the plane $z = 1$ of the xyz-space ξ and η correspond to coordinates x and y of that plane. Rotational conical flow is described by Equations (1.1) [1]:

$$L_1 = (u - \xi w)\left(\frac{u^2 + v^2 + w^2}{2}\right)_\xi + (v - \eta w)\left(\frac{u^2 + v^2 + w^2}{2}\right)_\eta + a^2(\xi w_\xi + \eta w_\eta - u_\xi - v_\eta) = 0 \qquad (1.1)$$

$$L_2 = (u - \xi w) s_\xi + (v - \eta w) s_\eta = 0$$

$$L_3 = \xi[(u - \xi w) u_\xi + (v - \eta w) u_\eta] + \eta[(u - \xi w) v_\xi + (v - \eta w) v_\eta] + (u - \xi w) w_\xi + (v - \eta w) w_\eta = 0$$

$$L_4 = w\{\xi[(u - \xi w) u_\eta - (v - \eta w) u_\xi] + \eta[(u - \xi w) v_\eta - (v - \eta w) v_\xi] + (u - \xi w) w_\eta - (v - \eta w) w_\xi\} + [(u - \xi w)^2 + (v - \eta w)^2](u_\eta - v_\xi) + a^2[(u - \xi w) s_\eta - (v - \eta w) s_\xi] = 0$$

These equations represent a combination of the equations of momentum, continuity and energy. Here c_v is the specific heat for constant volume, γ is the adiabatic index, a is the speed of sound:

$$s = S\,[(\gamma - 1)\,\gamma c_v]^{-1}, \qquad a^2 = a_1{}^2 - \frac{\gamma - 1}{2}(u^2 + v^2 + w^2 - W_1{}^2)$$

For irrotational flow the conical potential is

$$F(\xi, \eta) = z^{-1}\varphi(x, y, z)$$

Here ϕ, the velocity potential, satisfies the equation [2]

$$\{a^2(1 + r^2) - [rF - (1 + r^2)F_r]^2\}F_{rr} + 2\left[F - \left(r + \frac{1}{r}\right)F_r\right]F_\theta \times$$
$$\times \left(\frac{1}{r}F_{r\theta} - \frac{1}{r^2}F_\theta\right) + \left(a^2 - \frac{1}{r^2}F_\theta{}^2\right)\left(\frac{1}{r^2}F_{\theta\theta} + \frac{1}{r}F_r\right) = 0 \tag{1.2}$$
$$(r = \sqrt{\xi^2 + \eta^2},\ \tan\theta = \eta / \xi)$$

Equation (1.2) is of elliptic (hyperbolic) type if the projection of the velocity vector on the plane perpendicular to the radius-vector at a point in the xyz-space is smaller (larger) than the local velocity of sound.

From the form of the equations $L_2 = 0$, $L_3 = 0$ in the system (1.1) it follows immediately that the "streamlines" (line of constant S) are determined by the equation

$$\frac{d\xi}{u - \xi w} = \frac{d\eta}{v - \eta w} \tag{1.3}$$

yielding two characteristics of the system (1.1). The remaining two characteristics of the system (1.1), as one can show, coincide with those of Equation (1.2).

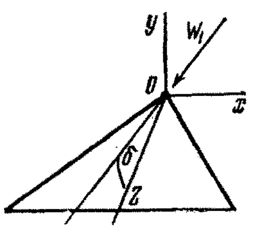

Fig. 1.

We now consider the picture of the flow about a triangular wing in the ξ, η-plane. Because of symmetry we need only illustrate the half of the flow for $\xi > 0$. The generally accepted scheme of the flow about the wing is given in Fig. 2 [3]. The wing is represented by the segment $O - 3$ (the z-axis lies in the plane of the wing). The enveloping Mach cones of the unperturbed flow, with vertices on the side edges of the wing, appear as the arcs $1 - 2$ of the Mach cone with the vertex in the apex point of the wing (point O on Fig. 1) and the segment of straight line $2 - 3$. For the flow about a sharp side edge there will be a Prandtl-Meyer flow which is extended until the velocity vector becomes parallel to the plane of the wing. This flow has a bundle of straight-line characteristics of Equation (1.2) going through point *3*. The segment *3* – *6* represents the boundary of the Prandtl-Meyer flow, after which there follows a singular flow which adjoins the surface of the wing. On the boundary of the common conical flow is the shock wave *2* – *7*, which passes near the curvilinear characteristic *2* – *6* of the Prandtl-Meyer flow, the straight-line characteristic *5* – *6*, and the arc of the Mach cone *5* – *4*. For the flow under the wing there will be on the lateral edge a plane shock *3* – *9*. After this follows a homogeneous flow. The region of the conical flow is bounded with a curvilinear shock *9* – *10* and an arc *9* – *8* of the Mach cone of the homogeneous flow behind the shock *3* – *9*.

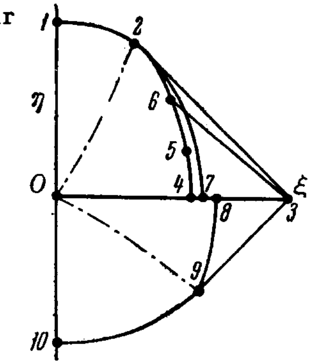

Fig. 2.

"Possible" shocks were introduced in [3] that lie close to the Mach cones *1* – *2*, *9* – *8*, and with regard to these it was not clear whether or not it is possible to do without them in a mathematical formulation. In this paper the author refrains from introducing similar shocks. In what follows, consideration is given, to begin with, to the structure of the flow in the region of the point *9* (Fig. 2), the joining of the irrotational flow behind the Mach cone *9* – *8*, and the rotational flow behind the curvilinear shock *9* – *10*. It appears that the joining of these flows without a singular point is impossible.

We take the coordinate axes so that the axis Oz is along the direction of the velocity of the homogeneous flow in the region *3* – *8* – *9* and the shock *3* – *9* is represented in the ξ, η-plane by a segment parallel to the ξ-axis (Fig. 3).

In this system of coordinates the Mach cone *8* – *9* (Fig. 2), is represented by the circular arc *8* – *9* (Fig. 3) of radius

$$r_0 = (M_0^2 - 1)^{-1/2}$$

where M_0 is the Mach number behind shock *3* – *9*: point *9* has the coordinates (ξ_0, η_0); the velocity vector of the unperturbed flow has components $(0, v_1, w_1)$; the velocity vector behind the shock *3* – *9* has components $(0, 0, w_0)$; and the polar coordinates of point *9* will be taken as $r = r_0$, $\theta = \theta_0$.

In [2], for the conical potential F in the neighborhood of the Mach cone *8* – *9*, the following expansion was found:

$$F = w_0 + \beta (r_0 - r)^2 + \gamma_1 (r_0 - r)^3 \ln (r_0 - r) + c(\theta)(r_0 - r)^3 + \cdots \quad (1.4)$$

Here

$$\beta = \frac{w_0}{2(\gamma+1)} \frac{(M_0^2-1)^2}{M_0^4}$$

$$\gamma_1 = \frac{w_0}{6(\gamma+1)^2} \frac{(M_0^2-1)^{5/2}}{M_0^6} [3(M_0^2-1)-(\gamma+1)(2-M_0^2)]$$

The function $c(\theta)$ is arbitrary; the dots denote terms of higher order in $r_0 - r$. If at point *9* there is no singularity, then the function $c(\theta)$ is regular at $\theta = \theta_0$ and the behavior of the solution in the neighborhood of point *9* is determined by the second term in the expansion (1.4); in particular, $F_r = F_\theta = F_{r\theta} = F_{\theta\theta} = 0$, $F_{rr} = 2\beta$. Using the formulas

$$u = \cos\theta F_r - \frac{1}{r}\sin\theta F_\theta, \qquad v = \sin\theta F_r + \frac{1}{r}\cos\theta F_\theta, \qquad w = F - rF_r \tag{1.5}$$

we find at point *9* the derivatives of u, v, w along the streamline *9 – 0*:

$$\begin{aligned} r_0 u_r &= \xi_0 u_\xi + \eta_0 u_\eta = \xi_0 2\beta \\ r_0 v_r &= \xi_0 v_\xi + \eta_0 v_\eta = \eta_0 2\beta \\ r_0 w_r &= \xi_0 w_\xi + \eta_0 w_\eta = -r_0^2 2\beta \end{aligned} \tag{1.6}$$

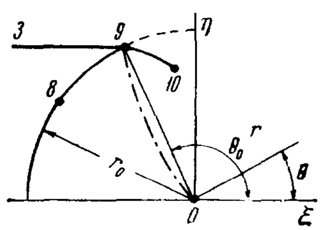

Fig. 3.

In addition $\xi_0 s_\xi + \eta s_\eta = 0$ because the flow is irrotational. If one writes the equations for the coefficients of the discontinuity of the derivatives of velocity components and s along the streamline *9 – 0* (entropy characteristic of the system (1.1)), then it is possible to convince oneself that for these equations point *9* does not have a singularity. It therefore follows that the rotational flow behind the shock *9 – 10* must at point *9* possess a finite derivative. The conditions on the conical shock can be written in the form [4]

$$u = u_1 - \eta' P, \qquad v = v_1 + P, \qquad w = w_1 + (\xi\eta' - \eta)P \tag{1.7}$$

$$\eta' = \frac{d\eta}{d\xi}, \qquad P = \frac{2}{\gamma+1}\left[\frac{a_1^2}{v_1 - \eta' u_1 + w_1(\xi\eta' - \eta)} - \frac{v_1 - \eta' u_1 + w_1(\xi\eta' - \eta)}{1 + \eta'^2 + (\xi\eta' - \eta)^2}\right]$$

where (u_1, v_1, w_1), (u, v, w) are the velocity components before and after the shock, respectively, a_1 is the velocity of sound before the shock, and $\eta = \eta(\xi)$ is the equation of the shock.

In addition, the jump in s is determined by the jump in entropy S from the formula

$$\Delta s = \frac{1}{\gamma(\gamma-1)}\left\{\ln\left[\frac{2\gamma q_n^2-(\gamma-1)}{\gamma+1}\right]+\gamma\ln\left[\frac{2+(\gamma-1)q_n^2}{(\gamma+1)q_n^2}\right]\right\} \tag{1.8}$$

where

$$q_n^2a_1^2 = [v_1-u_1\eta'+w_1(\xi\eta'-\eta)]^2[1+\eta'^2+(\xi\eta'-\eta)^2]^{-1}$$

Taking into account that for the shock *3 - 9* (Fig. 3) we have $u_1 = \eta' = 0$, from (1.7) we obtain the relations

$$w_0 = w_1+\eta_0 v_1, \qquad \left[a_1^2+v_1\frac{\gamma+1}{2}(v_1-\eta_0 w_1)\right](1+\eta_0^2)-(v_1-\eta_0 w_1)^2 = 0 \tag{1.9}$$

Writing the equation of the shock *9 - 10* in the neighborhood of point *9* in the form $\eta = \eta_0 - l(\xi-\xi_0)^2+\ldots$, where l is a constant, we get from (1.7), (1.8) the values of u, v, w, and s along the shock *9 - 10*. Differentiating with respect to ξ and taking into account the finite derivative of u, v, w and s with respect to ξ and η, we obtain at point *9*

$$u_\xi = -2lv_1, \quad v_\xi = l\frac{4\xi_0}{\gamma+1}\frac{w_1[(1+\eta_0^2)a_1^2+(v_1-\eta_0 w_1)^2]+2\eta_0(v_1-\eta_0 w_1)^3}{(v_1-\eta_0 w_1)^2(1+\eta_0^2)^2} \tag{1.10}$$

The conditions for joining irrotational and rotational flow along the streamline *0 - 9* are contained in an equation for the derivatives of u, v, w, s along *0 - 9*. Because the characteristic *0 - 9* of the system (1.1) is double, it is sufficient to require only the relations between the derivatives of u and v. The equation $L_1 = 0$ of (1.1) with consideration of (1.6) at point *9* can be represented in the form

$$u_\xi + v_\eta = 2\beta \tag{1.11}$$

Multiplying (1.11) by η_0 and subtracting from the second equation of (1.6) we obtain

$$\xi_0 v_\xi - \eta_0 u_\xi = 0 \tag{1.12}$$

Inserting (1.10) into (1.12) and expressing ξ_0, w_0, M_0 in terms of v_1, w_1, and η_1 after cancelling out l, we obtain

$$\begin{aligned}&\left[a_1^2(1+\eta_0^2)-\eta_0^2\frac{\gamma+1}{2}(w_1+\eta_0 v_1)^2+\frac{\gamma-1}{2}(v_1-\eta_0 w_1)^2\right]\times\\ &\times\{w_1(1+\eta_0^2)[(1+\eta_0^2)a_1^2+(v_1-\eta_0 w_1)^2]+2\eta_0(v_1-\eta_0 w_1)^3\}+\frac{\gamma+1}{2}\eta_0 v_1\times\\ &\times\left[\frac{\gamma+1}{2}(w_1+\eta_0 v_1)^2-\frac{\gamma-1}{2}(v_1^2+w_1^2)-a_1^2\right](v_1-\eta_0 w_1)^2(1+\eta_0^2)^2 = 0\end{aligned} \tag{1.13}$$

Substituting the actual values of $\gamma = 1.4$, $v_1 = 1.14497\ a_1$, $w_1 = 3.29694\ a_1$, satisfying (1.9), into (1.13) we obtain not zero but 3.65; this shows that (1.13) is not a consequence of (1.9), that is, the smooth joining of flows is impossible. We note that the result remains in force if F_{rr} at point *9* (Fig. 3) is an arbitrary constant and $F_{r\theta} = F_{\theta\theta} = 0$. The value $l = 0$ is also not taken on.

The question of the joining of rotational and irrotational conical flow is generally analogous to plane supersonic gas flow, if the solution (1.2) at the point of juncture is of hyperbolic type. We consider, for instance, the flow about the symmetrical profile (Fig. 4a) and a symmetrical triangular wing (Fig. 4b) in supersonic flow with the conditions that the edges of the conical wing are supersonic, the wings have a wedge-like form in the neighborhood of the leading edge, and the angle of attack is equal to zero. By virtue of symmetry the flow can be qualitatively represented in the neighborhood of the leading edge for η and $y > 0$.

Near the plane parts of the wings *0 – 1* there are plane shocks *0 – 2* behind which there follow homogeneous flows which attach themselves to the wing surfaces (region *0 – 1 – 2*). The curvature of the wings, beginning with point *1*, leads to the curvature of the shock *2 – 5* and the formation of rotational flows. These combine along the streamline *2 – 4* with the irrotational flow. Lines *1 – 2* and *2 – 3* are characteristics.

In the case of plane flow the desired function can have as velocity components u, v, and $s = S[c_v\gamma(\gamma - 1)]^{-1}$, where S is the entropy and the equations for these can be written in a form analogous to (1.1)

$$L_5 = v(u_y - v_x) - a^2 s_x = 0, \qquad L_6 = us_x + vs_y = 0$$
$$L_7 = (a^2 - u^2)u_x - uv(u_y + v_x) + (a^2 - v^2)v_y = 0 \tag{1.14}$$

The condition of the matching of rotational and irrotational flow at point *2* (Fig. 4a) appears as the coincidence of derivatives of u, v and s along the streamline (*2 – 4*) (the entropy characteristic). After satisfying the conditions on the shock wave (*2 – 5*) and the characteristic *2 – 3* there remain two undetermined parameters: the shock curvature at point *2* and the normal derivative of u or v on the characteristic *2 – 3* which makes it possible to equate the derivatives of u and v along *2 – 4* in rotational and irrotational flow. In conical flow the desired functions are u, v, w, s.

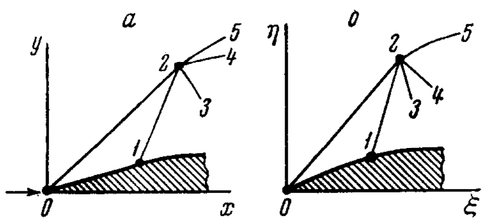

Fig. 4.

Here the situation is analogous to the plane case, with the only difference being that the streamline (*2* – *4*) (Fig. 4b) appears as a double characteristic of the system (1.1) guaranteeing the closing of both s and w. For the case of flow behind the Mach cone *8* – *9*, (Fig. 3), the situation is more complicated, since the Mach cone appears simultaneously as a characteristic and as a parabolic line of Equation (1.2). This leads to a special structure for the solution in this neighborhood and the impossibility of combining rotational and irrotational flows without singular points. (There will be an analogous result in the case where the wing is curved. In this case the Mach cone *8* – *9* (Fig. 2) is replaced by a characteristic curve which comes out from the parabolic point *9*.) The author was unable to find such a singular point which would do away with the difficulties of matching the flows at point *9* (Fig. 2) (see section 2). He has therefore considered other constructions for the flow at point *9*. At the present time, it has been possible to construct only one scheme which removes the difficulties at point *9*. (Fig. 5).

Point *9* is shifted along the shock wave *3* – *9* and takes up the new position *9′* (Fig. 5). Through point *9′* passes an additional weak shock wave *9′* – *11*, which dies out at point *11*, which itself lies on the Mach cone. Behind the shock *9′* – *11* there is a local Prandtl-Meyer flow beginning with the characteristic *9′* – *12* and ending with the characteristic *9′* – *13*, which assures the matching of the flows on a line of contact discontinuity *9′* – *0* (for the matching of flows with contact discontinuities one requires as many undetermined parameters as for the entropy characteristic *9* – *0*).

The curves *9′* – *12*, *9′* – *14* are characteristics behind the shock *9′* – *11*, the lines *9′* – *16*, *9′* – *15* are characteristics of the homogeneous flow, up to the shock, *9′* – *10* is a shock. The strength of the shock *9′* – *11* at point *9′* depends on the degree of expansion of the Prandtl-Meyer flow; if there is no expansion then the shock *9′* – *11* degenerates into a characteristic. The author investigated the asymptotic behavior of this scheme for an infinitely weak shock *3* – *9′* ($v_1 \to 0$) and found that the magnitude of the contact discontinuity is of higher order than v_1. Replacing the shock *9′* – *11* with a rectilinear characteristic is impossible, because the simple wave bordering it cannot be connected with the flow behind the shock *9′* – *10*.

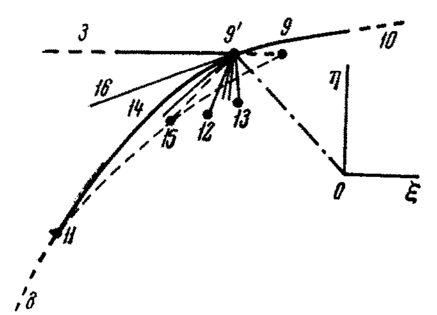

Fig. 5.

It is also impossible to join $9'$ with point 9, because in that case the shock $9' - 10$ would be a rarefaction discontinuity for $v_1 \to 0$. The shock waves $3 - 9'$, $9' - 10$, $9' - 11$ are incoming, but they are of different type. The shock $3 - 9'$ is plane, shocks $9' - 10$, $9' - 11$ are of the shock type generated in axisymmetric flow about a circular cone. We note that the construction of the flow of the type at point $9'$ is also met in the supersonic flow about a double wedge [5]. And so, the possible scheme of the flow on the bottom of the triangular wing is illustrated in Fig. 6 (with notations corresponding to Fig. 2, 5). The extension of the shock $9' - 11$ is small and tends to zero when the angle of attack tends to zero.

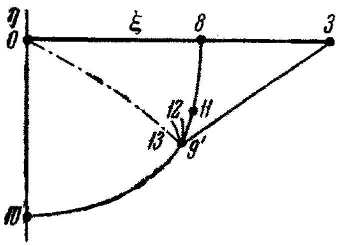

Fig. 6.

2. We shall now occupy ourselves with the construction of the solution in the neighborhood of point 11 (Fig. 6) and point 2 (Fig. 2). In this neighborhood begins the joining of the flow behind the Mach cone with the decaying shock wave. We shall be interested only in the leading terms of the solution in the neighborhood of these points and their structure. We shall start from point 11. Such a point also occurs in the flow about a triangular wing with subsonic edges, at the edge of a rectangular plate and so on, when the shock wave produced by the body weakens and transforms into the Mach cone which partially bounds the region. There is a perturbation of the flow about the body, on that side where there is an expansion of the undisturbed flow. We take the coordinate axes xyz so that the axis Oz will be directed parallel to the flow velocity on the Mach cone $8 - 11$ (Fig. 6) and the point 11 will have coordinates $(0, \eta_0)$ (Fig. 7). The Mach cone $8 - 11$ can therefore be represented by the circumference $r_0 = \eta_0 = (M_0^2 - 1)^{-1/2}$ and $\theta = \theta_0 = \pi/2$ at point 11.

We will start from the expansion (1.4) of the solution in the neighborhood of the Mach cone. A solution is sought with a singularity at point 11 so we assume

$$c(\theta) = \frac{\nu}{\vartheta^p} + \frac{\mu}{\vartheta^k} + \ldots, \qquad (\vartheta = \theta - \theta_0,\ p > k > 0) \qquad \left(\begin{array}{l}\nu, \mu - \text{arbitrary} \\ \qquad\ \text{constants}\end{array}\right)$$

Designating $\rho = (r_0 - r)\vartheta^{-k}$ and substituting $c(\theta)$ and $r_0 - r = \rho\,\vartheta^k$ in (1.4), we obtain

$$F = w_0 + (\nu\rho^3 + \ldots)\,\vartheta^{3k-p} + (\beta\rho^2 + \mu\rho^3 + \ldots)\,\vartheta^{2k} + \ldots$$

that is, the solution has to be sought in the form

$$F = w_0 + R(\rho)\,\vartheta^{3k-p} + F(\rho)\,\vartheta^{2k} + \ldots \tag{2.1}$$

where for small ρ

$$R(\rho) = \nu\rho^3 + \ldots, \qquad F(\rho) = \beta\rho^2 + \mu\rho^3 + \ldots$$

Substituting (2.1) in (1.2) and comparing coefficients of like powers of ϑ, we obtain $p - 2$, $1 < k < 2$.

$$\Omega(\rho) = R(\rho)\,c_1 \qquad \left(c_1 = \frac{(1 + r_0^2)\,w_0 r_0^3\,(\gamma + 1)}{a_0^2}\right)$$

satisfying the equation

$$(k^2\rho^2 - \Omega')\,\Omega'' - 5k\,(k - 1)\,\rho\Omega' + 3\,(k - 1)\,(3k - 2)\,\Omega = 0$$

and for small ρ

$$\Omega\;(\rho) = \tfrac{1}{3}\,\rho^3 + c_2\rho^{3 + \frac{1}{k-1}} + \ldots$$

where c_2 is an arbitrary constraint. These solutions correspond to the flow along bent walls which adjoin the Mach cone. Thus, for instance, with $k = 3/2$, we get the flow about wall with finite curvature.

The solution in that case has the form

$$\Omega(\rho) = c^{-3}v(t), \qquad t = c\rho$$

$$v\;(t) = -\tfrac{9}{10}\,[(\sqrt{1 + t^3} + t)^{1/3} - (\sqrt{1 + t^3} - t)^{1/3}]^3 + \\ + \tfrac{27}{20}t\,[(\sqrt{1 + t^3} + t)^{1/3} - (\sqrt{1 + t^3} - t)^{1/3}]^2$$

Here c is a constant, depending on the curvature of the wall at the Mach cone.

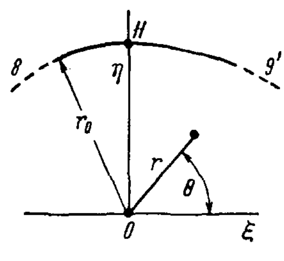

Fig. 7.

We shall now put $\nu \equiv 0$, that is, $R(\rho) = 0$. Thus from (1.2) we obtain the condition $k \geqslant 2$. We consider the cast $k > 2$ (as we shall see, it corresponds to the singularity at point *11*). We select further terms of the expansion of $c(\theta)$ in the form

$$c\,(\theta) = \frac{\mu}{\vartheta^k} + \frac{\lambda}{\vartheta^2} + \varepsilon\ln\vartheta + \nu_1 + \ldots$$

where μ, λ, ϵ, ν_1 are arbitrary constants.

Thus the corresponding solution will have the form

$$F = w_0 + F(\rho)\,\vartheta^{2k} + \Phi(\rho)\,\vartheta^{3k-2} + T(\rho)\vartheta^{3k}\ln\vartheta + R_1(\rho)\vartheta^{3k} + \ldots \tag{2.2}$$

where for small ρ

$$F(\rho) = \beta\rho^2 + \mu\rho^3 + \ldots, \qquad \Phi(\rho) = \lambda\rho^3 + \ldots$$

$$T(\rho) = (\gamma_1 k + \varepsilon)\rho^3 + \ldots, \quad R_1(\rho) = \gamma_1\rho^3 \ln\rho + \nu_1\rho^3 + \ldots$$

Substituting (2.2) into (1.2) and comparing the coefficients with similar powers of ϑ and ln ϑ, we obtain for F, Φ, T, R_1 the equations

$$F''[2r_0^2(w_0^2 - a_0^2)\rho - (1 + r_0^2)w_0 r_0^2(\gamma + 1)F'] - a_0^2 F' = 0 \qquad (2.3)$$

$$\Phi''[2r_0^2(w_0^2 - a_0^2)\rho - (1 + r_0^2)w_0 r_0^2(\gamma + 1)F'] - (1 + r_0^2)w_0 r_0^2(\gamma + 1)F''\Phi' -$$
$$- a_0^2\Phi' = -\frac{a_0^2}{r_0}[2k(2k - 1)F - k(3k - 1)\rho F' + k^2\rho^2 F''] \qquad (2.4)$$

and so on. These equations can easily be integrated in closed form and their solution satisfies the conditions for small ρ, for instance

$$F(\rho) = \frac{2\beta}{3c^2}[(1 + 2c\rho)^{3/2} - 3c\rho - 1]$$

where $c > 0$ is an arbitrary constant.

With large ρ we obtain asymptotic expressions of the form

$$F(\rho) = \frac{8\beta}{3\sqrt{2c}}\rho^{3/2} + \ldots, \qquad \Phi(\rho) = -\frac{2\beta}{15\sqrt{2c}} \cdot \frac{k(k-2)}{r_0}\rho^{5/2} + \ldots \qquad (2.5)$$

$$T(\rho) \sim \rho^{3/2}, \qquad R_1(\rho) \sim \rho^{5/2} \quad \text{or} \quad \rho^3, \quad \text{if} \quad k = 4$$

Changing over for large ρ in (2.2) to the variables ϑ and $\zeta = (r_0 - r)r_0^{-1}\vartheta^{-2}(\rho = \zeta r_0 \vartheta^{2-k})$ we have

$$F = w_0 + g[\zeta^{\frac{3}{2}} - \frac{k(k-2)}{20}\zeta^{\frac{5}{2}} + \ldots]\vartheta^{\frac{k}{2}+3} + \ldots \quad \left(g = \frac{8\beta r_0^{3/2}}{3\sqrt{2c}} > 0\right) \qquad (2.6)$$

The terms containing ζ are obtained from $F(\rho)$ and $\Phi(\rho)$. From (2.6) it follows that with $\rho \to \infty$ and finite ζ the solution has the form

$$F = w_0 + \chi(\zeta)\vartheta^{\frac{k}{2}+3} + \ldots \qquad (2.7)$$

where for small ζ

$$\chi(\zeta) = g[\zeta^{3/2} - \frac{k(k-2)}{20}\zeta^{5/2} + \ldots] \qquad (2.8)$$

Substituting (2.7) into (1.2) we get for $\chi(\zeta)$ the equation

$$2\zeta(1 + 2\zeta)\chi'' - [1 + 2\zeta(k + 3)]\chi' + \left(\frac{k}{2} + 2\right)\left(\frac{k}{2} + 3\right)\chi = 0 \qquad (2.9)$$

(This equation leads to the hypergeometric equation by replacing $2\zeta = -t$.) Substituting (2.8) into (2.9) we convince ourselves that (2.8) is the solution of (2.9) for small ζ. Moreover, for $k = 0, 2, 4, 6$ the series (2.8) terminates and the solution can be obtained in closed form

$$\chi(\zeta) = g\left[\zeta^{3/2} - \frac{k(k-2)}{20}\zeta^{5/2}\right] \tag{2.10}$$

For other k it is easy to find asymptotic expressions for χ for large ζ. Starting with (2.9) we have

$$\chi(\zeta) \sim \zeta^{\frac{k}{4}+\frac{3}{2}} \quad \text{or} \quad \chi(\zeta) \sim \zeta^{\frac{k}{4}+1} \tag{2.11}$$

We consider the cases $k = 4$, $k = 6$. The question now arises whether, to obtain the leading term of these solutions in the quadrant neighboring point *11*($\vartheta > 0$, $r_0 - r > 0$), the three first terms in the expansion (2.2) are to be retained:

$$F = w_0 + F(\rho)\vartheta^{2k} + \Phi(\rho)\vartheta^{3k-2}$$

For the approach to point *11* along the parabola $r_0 - r = \rho\vartheta^k$, $r_0 - r = \zeta r_0\vartheta^2$ (where ρ, ζ are fixed) the answer evidently is positive; investigation must be made only for $\zeta \to \infty$.

We transform in (2.7) to $r_0 - r$ and ϑ in place of ζ and ϑ; for $k = 4$ we obtain

$$F = w_0 + h\left[\vartheta^2(r_0 - r)^{3/2} - \frac{2}{5r_0}(r_0 - r)^{5/2}\right] + \ldots \quad \left(h = \frac{8\beta}{3\sqrt{2c}}\right) \tag{2.12}$$

Substituting (2.12) into (1.2), we easily convince ourselves that (2.12) is the leading term as one approaches point *11* on an arbitrary path. For $k = 6$ we obtain

$$F = w_0 + h\left[(r_0 - r)^{3/2}\vartheta^3 - \frac{6}{5r_0}(r_0 - r)^{5/2}\vartheta\right] + \ldots \tag{2.13}$$

Substituting (2.13) in (1.2), we find that (2.13) is the leading term only for an approach to point *11* on the parabola

$$\vartheta = \text{const}\,(r_0 - r)^{\alpha}, \qquad \alpha < \tfrac{5}{2}$$

Therefore it is necessary in the expansion (2.7) to take another term

$$F = w_0 + \chi(\zeta)\vartheta^6 + \Psi(\zeta)\vartheta^{10} + \ldots \tag{2.14}$$

The equation for $\Psi(\zeta)$ is easily integrable. Writing out the asymptotic representation of Ψ for $\zeta \to \infty$

$$\Psi(\zeta) = g^2 \frac{w_0}{a_0^2} \frac{18}{5} \zeta^4 + \dots \tag{2.15}$$

Transferring in (2.14) to $r_0 - r_1$, ϑ for $\zeta \to \infty$, we obtain

$$F = w_0 + h\left[(r_0 - r)^{3/2} \vartheta^3 - \frac{6}{5r_0}(r_0 - r)^{5/2} \vartheta\right] + h^2 \frac{w_0}{a_0^2 r_0} \frac{18}{5} (r_0 - r)^4 \vartheta^2 + \dots \tag{2.16}$$

Substituting (2.16) into (1.2), we assure ourselves that the terms written out in (2.16) give the leading term of the solution for an arbitrary law of approach to the point *11*. Moreover, for F_r and F_θ the leading term is given by the term with χ, and the term with Ψ gives the leading term for $F_{\theta\theta}$. And so, for the case $k = 6$, to the first three terms in (2.2) it is necessary to add other terms which give $\Psi(\zeta)$ in the transformation from ρ to ζ. At the same time it is important that they are all generated by only the first two terms $w_0 + F(\rho)\,\vartheta^{2k}$ in the expansion (2.2). Indeed, the term $\Phi(\rho)\,\vartheta^{3k-2}$ is related in the expansion (2.2) to $F(\rho)\,\vartheta^{2k}$; together they give $\chi(\zeta)$ for $\rho \to \infty$, and the term $\chi(\zeta)\,\vartheta^6$ in the expansion (2.14) calls for the appearance of the term $\Psi(\zeta)\,\vartheta^{10}$.

It follows that the leading term $w_0 + F(\rho)\,\vartheta^{2k}$ for $\rho = \text{const}$ leads to the leading term (2.16) in the neighborhood of the line $\vartheta = 0$. This brings to attention the non-self-similarity of the leading term of the solution in the neighborhood of point *11*. For the case $k \neq 4$, 6 when the analogous situation is examined it is possible to establish with the help of (2.11) that

$$\chi \sim \zeta^{\frac{k}{4} + \frac{3}{2}}, \qquad \chi \sim \zeta^{\frac{k}{4} + 1}$$

corresponding to the cases $k = 4$ and $k = 6$.

We utilize the case $k = 6$ for the construction of the solution in the neighborhood of point *11* (Fig. 7). We shall write the equation of the shock wave *11* – *9'* in the neighborhood of point *11* in the form $r_0 - r = \rho_0 \vartheta^6 + \dots$, where $\rho_0 < 0$ is an unknown constant.

Then the condition on the shock (1.7) yields

$$s = d_0 \vartheta^{18} + \dots, \qquad d_0 = -\frac{16}{3} \frac{\rho_0^3}{(\gamma+1)^3} \frac{(M_0^2 - 1)^{9/2}}{M_0^6} \tag{2.17}$$

and in the notation of the irrotational flow the condition for the velocities

$$F = w_0 \qquad F_r = -\frac{4w_0}{\gamma+1} \frac{(M_0^2 - 1)^2}{M_0^4} \rho_0 \vartheta^6 + \dots \tag{2.18}$$

We shall seek the irrotational solution in the neighborhood of the point *11*, satisfying the conditions on the Mach cone and the condition (2.18) on the shock wave. The leading term of this solution will be, as is shown below, the leading term of the solution for the velocity of the rotational problem. We seek F in the neighborhood of the shock *11* – *9′* in the form of the expansion

$$F = w_0 + F_1(\rho)\vartheta^{2k} + \Phi_1(\rho)\vartheta^{3k-2} + \ldots, \qquad \rho = (r_0 - r)\vartheta^{-k}, \qquad k = 6 \tag{2.19}$$

that is, the form of the expansion is the same as in the neighborhood of the Mach cone, only the arbitrary constants in $F_1(\rho)$ have to be determined from (2.18). For $F_1(\rho)$ we obtain

$$F_1(\rho) = -\frac{2\beta}{c_1}\left[\frac{(1+2c_1\rho)^{3/2} - (1+2c_1\rho_0)^{3/2}}{3c_1} + \rho - \rho_0\right] \qquad \left(c_1 = \frac{3}{8\,|\rho_0|}\right)$$

$$\Phi_1(\rho) = \frac{2\beta}{15\sqrt{2c_1}}\frac{k(k-2)}{r_0}\rho^{5/2} + \ldots, \qquad F_1(\rho) = -\frac{8\beta}{3\sqrt{2c_1}}\rho^{3/2} + \ldots \quad \text{for } \rho \to \infty$$

Transforming to ϑ and ζ we obtain

$$F = w_0 + \chi_1(\zeta)\vartheta^{\frac{k}{2}+3} + \ldots$$

where

$$\chi_1(\zeta) = g_1\left[\zeta^{3/2} - \frac{k(k-2)}{20}\zeta^{5/2}\right], \qquad g_1 = h_1 r_0^{2/3} < 0, \qquad h_1 = -\frac{8\beta}{3\sqrt{2c_1}} < 0$$

For $\zeta \to \infty$, after the substitution $k = 6$, we obtain

$$F = w_0 + g_1\left[\zeta^{3/2} - \frac{6}{5}\zeta^{5/2}\right]\vartheta^6 + g_1^2\frac{w_0}{a_0^2 r_0}\frac{18}{5}\zeta^4\vartheta^{10} + \ldots$$

Transforming to $r_0 - r$, ϑ and taking into account that $\vartheta < 0$ for the shock, we obtain

$$F = w_0 - h_1\left[(r_0 - r)^{3/2}\vartheta^3 - \frac{6}{5r_0}(r_0 - r)^{5/2}\vartheta\right] + h_1^2\frac{w_0}{a_0^2 r_0}\frac{18}{5}(r_0 - r)^4\vartheta^2 + \ldots \tag{2.20}$$

Comparing (2.20) and (2.16) we find that (2.20) is an analytical continuation of (2.16) for $\vartheta < 0$, if $c_1 = c$. In the same manner, the leading term of the rotational solution is constructed, satisfying the condition on the Mach cone and the shock, and containing the undetermined parameter ρ_0, characterizing the curvature of the shock. Now we substitute the determined values of velocity in $L_2 = 0$ (1.1) and we determine s. Then we establish that the derivative of s in $L_4 = 0$ (1.1) will be small in comparison with the derivatives of the velocities. This means that the leading term of the solution for the velocities coincides for the rotational and irrotational problem. After the transformation to polar coordinates and after discarding the small terms, $L_2 = 0$ takes the form

$$s_r - \frac{1}{r_0^3 w_0} F_\theta s_\theta = 0 \qquad (2.21)$$

where for F_θ it is necessary to take the leading term in the form

$$F_\theta = \vartheta^{11}[12F_1(\rho) - 6\rho F_1'(\rho)] + \vartheta^{15}[16\Phi_1(\rho) - 6\rho\Phi_1'(\rho)]$$

After transformation to new independent variables ρ and $\vartheta = \theta - \theta_0$, (2.21) becomes

$$s_\rho[1 - \vartheta^{16} 6\rho A(\rho) - \vartheta^{20} 6\rho B(\rho)] + s_\vartheta[\vartheta^{17} A(\rho) + \vartheta^{21} B(\rho)] = 0 \qquad (2.22)$$

$$\left(A(\rho) = \frac{1}{r_0^3 w_0}[12F_1(\rho) - 6\rho F_1'(\rho)], \quad B(\rho) = \frac{1}{r_0^3 w_0}[16\Phi_1(\rho) - 6\rho\Phi_1'(\rho)]\right)$$

We seek the solution (2.22) in the form

$$s(\rho, \vartheta) = \sum_{n=1}^{\infty} s_n(\rho)\vartheta^{17+n}$$

where (2.17) gives $s_1(\rho_0) = d_0$. From the recursion relations for $s_n(\rho)$ it is easy to show that $s_1(\rho) \equiv d_0$ and the order of the growth of $s_n(\rho) \to \infty$ does not exceed $\rho^{n-25/4}$; from this it follows that, for $\rho \to \infty$ and finite $\zeta = \rho\vartheta^4 r_0^{-1}$, s is represented in the form

$$s = \sum_{n=0}^{\infty} p_n(\zeta)\vartheta^{18+n}$$

$$p_0(0) = d_0, \ p_1(\zeta) \sim \zeta^{5/2}, \ p_2(\zeta) \sim \zeta^5, \ p_3(\zeta) \sim \zeta^{15/2}, \ p_4(\zeta) \sim \zeta^{10}, \ldots, \text{ for } \zeta \to 0$$

After transforming to the independent variables ζ, ϑ instead of ρ, ϑ, Equation (2.22) is transformed into the form

$$s_\zeta[1 - 2(a\zeta^{5/2} + b\zeta^{7/2})\vartheta^6] + \vartheta^7(a\zeta^{5/2} + b\zeta^{5/2})s_\vartheta = 0 \qquad (2.23)$$

where a and b are determined from the relation

$$A(\rho) \sim a\rho^{3/2}, \qquad B(\rho) \sim b\rho^{5/2} \qquad \text{for } \rho \to \infty$$

With the help of the recurrence relation for $\rho_n(\zeta)$ obtained from Equation (2.23), it is possible to establish that for $\zeta \to \infty$ the order of growth of $\rho_n(\zeta)$ do not exceed $\zeta^{7n/2}$, from which it follows that for small r

$$s = \vartheta^{18}\sum_{n=0}^{\infty} d_n\tau^n \qquad (\tau = (r_0 - r)^{7/2}\vartheta^{-1}, \ d_n = \text{const})$$

This means that for finite r the solution is represented in the form

$$s = \vartheta^{18} Q(\tau) + \ldots, \qquad Q(0) = d_0$$

Transforming in (2.23) to the independent variables r and ϑ, we obtain

$$s_\tau \left(\tfrac{7}{2} + b\tau\right) - b s_\vartheta = 0$$

From here for $Q(r)$ we easily find

$$Q' \left(\tfrac{7}{2} + b\tau\right) - 18bQ = 0, \qquad Q(\tau) = d_0 \left(1 + \tfrac{2}{7} b\tau\right)^{18}$$

From this

$$s(r, \theta) = d_0 \left(1 + \tfrac{2}{7} b\tau\right)^{18} \vartheta^{18} + \ldots = d_0 \left[\vartheta + \tfrac{2}{7} b (r_0 - r)^{7/2}\right]^{18} + \ldots \quad (2.24)$$

From (2.24) it follows, in part, that the equation of the streamline $s = 0$ is

$$\vartheta = -\frac{2}{7} b (r_0 - r)^{7/2} = \ldots = -\frac{32}{35} \frac{\beta}{r_0^4 w_0 \sqrt{2c}} (r_0 - r)^{7/2} + \ldots$$

One can also obtain, by means of immediate integration of the equation defining the streamline (2.16)

$$\frac{d\vartheta}{dr} = -\frac{1}{r_0^3 w_0} F^\theta = \frac{16}{5} \frac{\beta}{r_0^4 w_0 \sqrt{2c}} (r_0 - r)^{5/2} + \ldots$$

The representation of s for finite ρ, ζ, r shows that the derivatives of s are small in comparison with the velocity derivatives in $L_4 = 0$.

We shall now concern ourselves with an investigation of the solutions in the region of the point 2 (Fig. 2). We take the system of coordinates in such a way that the z-axis is directed along the velocity on the Mach cone *1* – 2, and the characteristic 2 – *3* is taken to be parallel to the ξ-axis (Fig. 8). For an approach to point 2 from the left ($\xi = -0$) and from the right ($\xi = +0$) we have two different singularities. For the construction of the singularity for $\xi = -0$ we use the expansion (2.2) for $k = 2$. The variable ρ for $k = 2$ is $\rho = \zeta r_0$, and the equation for $Y(\zeta)$, which is defined by the relation $F(\rho) = 2\beta\, r_0^{\,2} Y(\zeta)$, will be written in the form

$$(2\zeta + 4\zeta^2 - Y') Y'' - (10\zeta + 1) Y' + 12Y = 0 \quad (2.25)$$

In the case $Y = 1/2\, \zeta^2 + \lambda \zeta^3 + \ldots$ ($\lambda = \text{const}$) for small ζ. Equation (2.25) cannot be integrated in closed form, but the character of the asymptotic behavior of Y for $\zeta \to \infty$ is not hard to determine. There are two possible types of asymptotic representations:

$$Y(\zeta) \sim a_1\zeta^2 + a_2\zeta^{3/2},$$

$$Y(\zeta) \sim \frac{\zeta^3}{3} + \frac{\zeta^2}{2} + a_3\zeta \quad \text{for } \zeta \to \infty \quad (a_1,\ a_2,\ a_3 = \text{const})$$

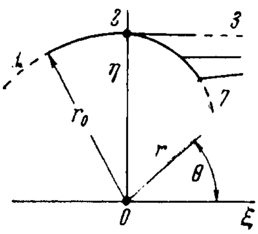

Fig. 8.

There exist also such λ that for some $\zeta < \infty$

$$|Y| < \infty, \quad |Y'| < \infty, \quad |Y''| = \infty$$

We are interested in the solutions with asymptotic behavior $Y \sim a_1\zeta^2 + a_2\zeta^3$. For these it is known that there exists at least one such solution $Y = 1/2\,\zeta^2$, and there evidently exist also other solutions of this type. On the streamline coming out of point $2(\zeta \to \infty)$ these solutions will give

$$u = -2\beta\frac{a_2}{r_0^2}(r_0 - r)^{3/2} + \ldots, \qquad v = -4\beta a_1(r_0 - r) + \ldots$$
$$w = w_0 - r_0 v + \ldots \tag{2.26}$$

For the construction of the solution for $\xi = +0$ we write the equation of the shock wave $2-7$ in the region of point 2 in the form $\eta = \eta_0 - l\xi^2 + \ldots$, where $\eta = r_0$ and l is an unknown constant. Utilizing the solution for the simple wave in the region of the parabolic line [4] from (1.7), we find the velocity components behind the shock $2-7$

$$u = -l^2\frac{16w_0}{\gamma+1}\frac{(M_0^2-1)^{3/2}}{M_0^4}\left(l - \sqrt{M_0^2-1}\right)\xi^3 + \ldots \tag{2.27}$$
$$v = -l\frac{2w_0}{\gamma+1}\frac{(M_0^2-1)^{3/2}}{M_0^4}\left(4l - 3\sqrt{M_0^2-1}\right)\xi^2 + \ldots$$
$$w = w_0 - \eta_0 v + \ldots, \qquad s = O(\xi^6)$$

We shall seek the solution behind the shock wave $2-7$ in the form

$$u = u(\sigma)\xi^3 + \ldots, \quad v = v(\sigma)\xi^2 + \ldots, \quad w = w_0 + w(\sigma)\xi^2 + \ldots,$$
$$s = s(\sigma)\xi^6 + \ldots, \quad \sigma = \frac{\eta_0 - \eta}{\xi^2}\eta_0$$

Substituting u, v, w, and s in the system (1.1), we find that the leading term of the solution will be irrotational and $u(\sigma)$, $v(\sigma)$, $w(\sigma)$ can be expressed in terms of one function $X(\sigma)$

$$u(\sigma)=\frac{2w_0}{\gamma+1}\frac{(M_0^2-1)^3}{M_0^4}(2X-\sigma X'),\qquad v(\sigma)=-\frac{w_0}{\gamma+1}\frac{(M_0^2-1)^{5/2}}{M_0^4}X'$$
$$w(\sigma)=-\eta_0 v(\sigma) \tag{2.28}$$

and the function $X(\sigma)$ satisfies the equation

$$(X'+2\sigma-4\sigma^2)X''+(10\sigma-4)X'-12X=0 \tag{2.29}$$

The conditions (2.27) yield that, for value $\sigma=\sigma_0$, corresponding to the shock wave

$$X(\sigma_0)=\sigma_0^2,\qquad X'(\sigma_0)=2\sigma_0(4\sigma_0-3)\qquad (0<\sigma_0<1) \tag{2.30}$$

For $\sigma\to\infty$ there are possible two types of asymptotic expressions:

$$X(\sigma)\sim b_1\sigma^2+b_2\sigma^{3/2},\qquad X(\sigma)\sim\frac{\sigma^3}{3}+b_3\sigma\qquad (b_1,\ b_2,\ b_3=\text{const})$$

For the problem under consideration it is appropriate to use the solution of the first type. In particular, Equation (2.29) has a family of exact solutions

$$X=\sigma^2+b_2\left(\sigma^{3/2}+\tfrac{3}{32}b_2\right).$$

which for $b_2=-32/27\sqrt{3}$ and $\sigma_0=1/3$ satisfies conditions (2.30). Evidently, there exist solutions with $b_1\neq 1$. On the streamline coming out of point 2 (Fig. 8) ($\sigma\to\infty$) these solutions give

$$u=b_2\frac{w_0}{\gamma+1}\frac{(M_0^2-1)^{5/2}}{M_0^4}(\eta_0-\eta)^{1/2}+\ldots,$$
$$v=-b_1\frac{2w_0}{\gamma+1}\frac{(M_0^2-1)^2}{M_0^4}(\eta_0-\eta)+\ldots,\qquad w=w_0-\eta_0 v+\ldots \tag{2.31}$$

Comparing (2.26) and (2.31) one can come to the conclusion that for the matching of the flows it is necessary that a_1, a_2, which depend on λ, and b_1, b_2, which depend on σ_0, have equal coefficients for like powers of $\eta_0-\eta$, r_0-r. This leads to a system of equations defining λ and σ_0, which, evidently, can be constructed and solved by the numerical integration of (2.25) and (2.29). It is then possible to make a final deduction about the possibility of the matching of the flows at point 2.

The singularities considered above, and more complicated ones, found by the author, do not solve the problem of the matching of the flows at point *9* (Fig. 3). We can only say that the solution of (1.1) behind the shock *9* – *10*, we have taken in the form $u=u(\phi)\,\epsilon^{\alpha}+\ldots$, $v=v(\phi)\,\epsilon^{\alpha}+\ldots$, $w=w_0+w(\phi)\,\epsilon^{\alpha}+\ldots$, where ϵ, ϕ are the polar coordinates on the

ξ, η, beginning in point 9. $a > 0$ is a constant.

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